PROPAGATION OF RADIAL CRACKS IN A ROUND BAR WITH TORSION

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The article considers the problem of brittle failure with the torsion of a cylindrical bar whose cross section is a circle of radius R, with an arbitrary number of radial divisions of length l. The problem is reduced to a form convenient for digital-computer computation. On the basis of the Griffith criterion, a determination is made of the value of the external load, corresponding to the start of the growth of a crack, as a function of the depth of the initial notches and their number.

1. Let us consider the problem of the torsion of a round bar, having the transverse cross section depicted on Fig. 1. We shall seek the solution by the methods of the theory of functions of a complex variable [1], using the conformal mapping of a circle with notches (Fig. 1) on the interior of a unit circle (Fig. 2).

In accordance with [1], the complex function of $f(\zeta)$ in the transformed region has the form

$$f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{\omega(\sigma)\,\overline{\omega(\sigma)}}{\sigma - \zeta}\, d\sigma \tag{1.1}$$

where γ is a unit circle; σ is a point of the contour; $\omega(\zeta)$ is the mapping function which, in the case under consideration, has the form [2]

$$z = (4)^{-1/n} (1 + \alpha) R\, [\zeta^{n/2} + \zeta^{-n/2} - \sqrt{(\zeta^{n/2} + \zeta^{-n/2})^2 - 4(1+\alpha)^{-n}}]^{4/n}$$
$$1 + \alpha = [(1 - a)^n + 1]^{2/n} / (4)^{1/n} (1 - a), \quad a = l / R \tag{1.2}$$

With mapping, the apexes of a crack, A_k, go over into the points of the unit circle a_k

$$|a_k| = 1, \quad \arg a_k = 2(k-1)\pi / n$$

The points of intersection of the circle with the notches go over into the points

$$|b_k| = 1, \quad |b_k'| = 1, \quad \arg b_k, \quad b_k' = \pm 2n^{-1} \arccos (1+\alpha)^{n/2} + 2(k-1)\pi / n$$

The points b_k, b_k' are the branch points of the function $z = \omega(\zeta)$. The single-valued branch of this function is selected from the condition for congruence of the boundaries. Writing σ and ζ in the form $\sigma = e^{i\theta}$, $\zeta = re^{i\varphi}$, and taking into account that, in the segments $|b_k, b_k'|$

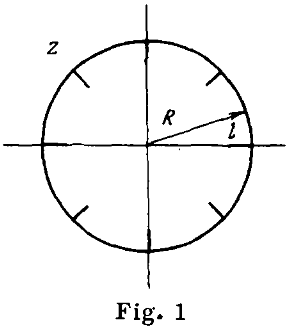

Fig. 1

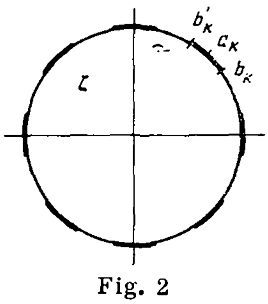

Fig. 2

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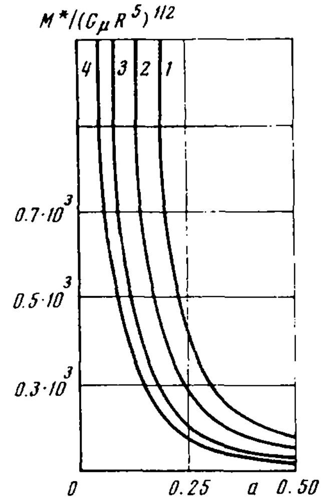

Fig. 3

$$\omega(\sigma)\,\overline{\omega(\sigma)} = 1 \tag{1.3}$$

we obtain the complex function of the torsion $f(re^{i\varphi})$ in the form

$$f(re^{i\varphi})\ \frac{(1+\alpha)^2}{\pi}\sum_{k=1}^{n}\int\limits_{-A+2(k-1)\pi/n}^{A+2(k-1)\pi/n}\frac{ie^{i\theta}\{\cos\gamma-\sqrt{\cos^2\gamma-(1+\alpha)^{-n}}\}^{4/n}d\theta}{e^{i\theta}-re^{i\varphi}}$$

$$+\frac{1}{2\pi}\sum_{k=1}^{n}\ln\left|\frac{-A+2(k-1)\pi/n-re^{i\varphi}}{A+2(k-1)\pi/n-re^{i\varphi}}\right|$$

$$A = 2/n^{-1}\arccos\left[(1+\alpha)^{-n/2}\right],\quad \gamma = \theta n/2+(k-1)\pi \tag{1.4}$$

To solve the torsion problem, it is necessary to calculate the rigidity

$$D = \mu(J + D_0) \tag{1.5}$$

where μ is the shear modulus; J is the polar moment of inertia of the area of a transverse cross section with respect to the center (in the given case, the polar moment of inertia $J = \pi R^4/2$), and the value of D_0 is calculated using the formula [1]

$$D_0 = -\frac{1}{4}\int\limits_{\gamma}\{f(\sigma)+\overline{f(\sigma)}\}\,d\{\omega(\rho)\overline{\omega(\sigma)}\} \tag{1.6}$$

which, taking account of relationships (1.3), (1.4), can be written in the form

$$D_0 = -Q\int\limits_{-A}^{A}\int\limits_{-A}^{A}\left\{\cos\frac{\varphi_1 n}{2}-\sqrt{\cos^2\frac{\varphi_1 n}{2}-(1+\alpha)^{-n}}\right\}^{4/n}\left\{\cos\frac{\theta_1 n}{2}-\sqrt{\cos^2\frac{\theta_1 n}{2}-(1+\alpha)^{-n}}\right\}^{4/n}d\theta_1 d\varphi_1\times$$

$$\times\left[\sqrt{\cos^2\frac{\varphi_1 n}{2}-(1+\alpha)^{-n}}\,\mathrm{tg}\,\frac{\theta_1-\varphi_1}{2}\right]^{-1}$$

$$Q = n(1+\alpha^4)/4\pi \tag{1.7}$$

In Eq. (1.7) we make the replacement of variables $\theta_1 = \theta+2(k-1)\pi/n$, $\varphi_1 = \varphi+2(k-1)\pi/n$ and we set $r = 1$ [passing to the limit under the double integral sign in Eq. (1.7) is admitted].

After certain transformations, Eq. (1.7) can be written in the form

$$D_0 = \frac{2(1+\alpha)^2}{\pi}\int\limits_{B}^{C}t^{4/n-1}\ln\left|\frac{\sin[A-\varphi(t)]/2}{\sin[A+\varphi(t)]/2}\right|dt + \frac{8(1+\alpha)^4}{\pi n}\int\limits_{B}^{C}\int\limits_{B}^{C}t^{4/n-1}u^{4/n-1}\ln\left|\frac{\sin[\varphi(t)-\varphi(u)]/2}{\sin[\varphi(t)+\varphi(u)]/2}\right|dtdu$$

$$B = 1-\sqrt{1-(1+\alpha)^{-n}},\qquad C = (1+\alpha)^{-n/2}$$

$$\varphi(x) = 2n^{-1}\arccos\,(x^2-(1+\alpha)^{-n})/2x$$

2. Let us consider the process of the propagation of cracks from the point of view of the energy concepts developed by Griffith [3].

Let all the notches receive small virtual increments δl, in their own planes (it is shown in [4] that, under conditions of torsion, a crack does not change its direction). Then, the equation of the energy balance existing with the growth of a crack is written in the form

$$\delta W/\delta l = G \tag{2.1}$$

Here W is the elastic energy accumulated inside a bar of unit length; G is a constant of the material, having the sense of the specific surface energy.

The elastic energy accumulated inside a bar of unit length with torsion is calculated using the formula [1]

$$W = M^2/2D \tag{2.2}$$

where M is the principal moment of the external stresses.

Substituting Eq. (2.2) into (2.1) and taking account of Eq. (1.5), we obtain

$$\delta W / \delta l = -M^2 (\partial D / \partial l) / 2D^2 \tag{2.3}$$

Using Eq. (2.3) we can determine the value of the critical external load M* corresponding to the start of the growth of a crack from a notch. It can be seen from Eq. (1.7) that M* will be a function of the depth of the initial notches and their number n

$$M^* = \sqrt{2D\,(G)}^{-1/2} / (-\partial D / \partial l)^{-1/2} \tag{2.4}$$

The function under the integral sign in both the first and second integrals has a singularity (in the first integral with t = C, and in the second with t = u).

Expression (1.8) was calculated using the formula

$$D_0 = \frac{2(1+\alpha)^2}{\pi} \int_B^{C-\varepsilon} t^{4/n-1} \ln \left| \frac{\sin(A-\varphi(t))/2}{\sin(A+\varphi(t))/2} \right| dt + \frac{8(1+\alpha)^4}{\pi n} \int_{B+\varepsilon}^{C} dt \int_B^{t-\varepsilon} t^{4/n-1} u^{4/n-1} \ln \left| \frac{\sin(\varphi(t)-\varphi(u))/2}{\sin(\varphi(t)+\varphi(u))/2} \right| du +$$
$$+ \frac{8(1+\alpha)^4}{\pi n} \int_B^{C-\varepsilon} dt \int_{t+\varepsilon}^{C} t^{4/n-1} u^{4/n-1} \ln \left| \frac{\sin(\varphi(t)-\varphi(u))/2}{\sin(\varphi(t)+\varphi(u))/2} \right| du$$

where $\varepsilon = 0.0001$. In this case, the error does not exceed 0.0012. The integrals were calculated using the Simpson formula.

The integral (1.8) was calculated in a Mir-1 digital computer. The dependence of the dimensionless critical load $M^* / (G\mu R^5)^{1/2}$ on the relative depth of the original notches l_0 and on their number is shown on Fig. 3. The curves designated by the numbers 1, 2, 3, and 4 correspond to a number of notches n = 2, 4, 6, and 7.

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